Baire measurable perfect matchings

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Given a (locally finite) Borel graph G on a Polish space $V(G)$, we study whether there is a Borel comeager invariant set $A \subseteq V(G)$ such that $G \restriction A$ admits a Borel perfect matching.

In this case I will say G admits a Baire measurable perfect matching.

Irrational rotations

Let $R : [0, 1) \to [0, 1)$ be an irrational rotation of the circle, i.e. $R(x) = x + \alpha \mod 1$ for some irrational α. Then the graph induced by R does not admit a Baire measurable perfect matching.

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Proof idea.

Use the fact that R^2 is generically ergodic, i.e. every Borel R^2-invariant set is either meager or comeager. □

Perfect matchings in finite bipartite graphs

Theorem (Hall, 1935)

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A compactness argument also yields:

Theorem

Let G be a locally finite bipartite graph, and suppose that $|N(F)| \geq |F|$ for all finite independent sets $F \subseteq V(G)$. Then G admits a perfect matching.

Baire measurable perfect matchings in bipartite Borel graphs

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Theorem (Marks-Unger, 2016)

Let G be a locally finite bipartite Borel graph satisfying $|N(F)| \geq (1+\varepsilon)|F|$ for all finite independent sets $F \subseteq V(G)$, for some fixed $\varepsilon > 0$. Then G admits a Baire measurable perfect matching.

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Let G be a locally finite bipartite Borel graph satisfying $|N(F)| \geq (1+\varepsilon)|F|$ for all finite independent sets $F \subseteq V(G)$, for some fixed $\varepsilon > 0$. Then G admits a Baire measurable perfect matching.

Corollary

Every bipartite regular non-amenable Borel graph admits a Baire measurable perfect matching.

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Every bipartite regular non-amenable Borel graph admits a Baire measurable perfect matching.

Definition

An infinite connected graph G of bounded degree is non-amenable if there exists $\delta > 0$ such that $|\partial F| \geq \delta |F|$ for all finite $F \subseteq V(G)$.

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If G is the Schreier graph of a free Borel action of a finitely generated non-amenable group, then G admits a Baire measurable perfect matching.

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Conjecture

If G is the Schreier graph of a free Borel action of a finitely generated group that is not 2-ended, then G admits a Baire measurable perfect matching.

Tutte's theorem

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Theorem (Tutte, 1950)

A locally finite graph G admits a perfect matching iff for all finite $X \subseteq V(G)$ we have

$$|X| \geq |\mathcal{C}_{\text{odd}}(G - X)|$$

Here $\mathcal{C}_{\text{odd}}(G - X)$ denotes the set of odd components of $G - X$.

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The Marks-Unger result generalized Hall's theorem to the Baire measurable setting by replacing "$|N(F)| \geq |F|$" with "$|N(F)| \geq (1 + \varepsilon)|F|$". Can we do the same for Tutte's theorem?

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Let G be a locally finite Borel graph, and suppose there exists $\varepsilon > 0$ such that for every finite set $X \subseteq V(G)$, we have

$$|X| \geq |\mathcal{C}_{\mathsf{odd}}(G - X)| + \varepsilon|\mathsf{hull}_{\mathsf{odd}}(X)|.$$

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Definition

$\mathrm{hull}_{\mathrm{odd}}(X) := X \cup \bigcup \mathcal{C}_{\mathrm{odd}}(G - X)$

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Definition

A graph G satisfies $\mathrm{Tutte}_{\varepsilon,k}$ if (a) Tutte's condition holds, and (b) whenever $X \subseteq V(G)$ is finite such that $\mathrm{hull}_{\mathrm{odd}}(X)$ is connected and $|\mathrm{hull}_{\mathrm{odd}}(X)| \geq k$, we have $|X| \geq |\mathcal{C}_{\mathrm{odd}}(G - X)| + \varepsilon|\mathrm{hull}_{\mathrm{odd}}(X)|$.

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Lemma (Marks-Unger, 2016)

Let G be a locally finite Borel graph, and let $f : \mathbb{N} \to \mathbb{N}$. Then there exist Borel sets $A_n \subseteq V(G)$ such that $\bigcup_n A_n$ is a Borel comeager invariant set and $d_G(x, y) > f(n)$ whenever x, y are distinct vertices in A_n.

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Check this works!

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Lemma

Let G be an (infinite, connected, locally finite) non-amenable vertex transitive graph. Then there exists $\varepsilon > 0$ such that for all finite $X \subseteq V(G)$, $|X| \geq |\mathcal{C}_{\mathrm{odd}}(G - X)| + \varepsilon|\mathrm{hull}_{\mathrm{odd}}(X)|$.

Proof sketch.

If d is the degree, and $\delta > 0$ is the expansion constant, then

$$\begin{aligned} d|X| &= \left|E\left(X, \bigcup \mathcal{C}_{\mathrm{odd}}(X)\right)\right| + \left|E\Big(X, V(G) \setminus \mathrm{hull}_{\mathrm{odd}}(X)\Big)\right| \\ &\geq d|\mathcal{C}_{\mathrm{odd}}(X)| + \delta|\mathrm{hull}_{\mathrm{odd}}(X)|. \end{aligned}$$

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Try to find more applications of the Marks-Unger proof technique using the above lemma, in the context of non-amenable/expansive graphs.

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Question

Does every non-amenable, $2d$-regular Borel graph have a Baire measurable Schreier decoration?

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Theorem (Lyons-Nazarov, 2011)

Let G be a locally finite bipartite pmp graph satisfying $\mu(N(A)) \geq (1 + \varepsilon)\mu(A)$ for all Borel independent sets A, for some fixed $\varepsilon > 0$. Then G admits a μ-measurable perfect matching.

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Theorem (Csóka-Lippner, 2017)

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Balanced orientations

Theorem (Bencs-Hrušková-Toth, 2021)

Every non-amenable, quasi-transitive, unimodular graph with all degrees even has a factor of i.i.d. balanced orientation.

A balanced orientation of a graph with all degrees even is an orientation for which each vertex has in-degree equal to out-degree.

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Question

What is the relationship between factor of i.i.d results and Baire measurable results, for non-amenable graphs?